

On making holes in a sheet of fluid

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It is suggested in this paper that axisymmetric holes in thin sheets of fluid in which surface tension forces predominate will open out if they are initially large in relation to the thickness of the sheet; but that small holes will close up. No exact criterion has been found for the critical hole size in a free falling sheet, but the behaviour of the sheet may be closely simulated by the suspension of a soap film between coaxial circular rings. Theoretical results and experimental observations on catenoid films so formed are described.

For a hole in a sheet standing under gravity on a horizontal plane an equilibrium configuration exists, which is shown to be unstable. It is suggested that in this case the equilibrium position serves to distinguish between holes which open and those which close. Experiments on the behaviour of holes in a mercury sheet reveal a well-defined critical size which is in good agreement with that predicted by the unstable equilibrium.

A further series of experiments on holes made in a sheet of water standing on paraffin wax gave no sharp distinction between opening and closing holes, and holes of a wide range of sizes could remain stationary. This behaviour is associated with changes in the angle of contact with the plane. Independent meniscus observations similar to those of Ablett for a steadily moving meniscus show that the angle of contact θ_a for a meniscus about to advance is greater than the value θ_r for a meniscus on the point of receding. It is seen that this difference will produce a range of hole diameters within which a hole will be trapped and remain stationary. Observations on the minimum size of hole on a water sheet which will remain open are reported. But it was found that the largest holes which would remain stationary were too large in relation to the size of the sheet for reliable results to be obtained.

1. Introduction

Studies of the disintegration of fluid sheets (Dombrowski & Frazer 1954) have shown that large holes expand, but the photographs sometimes reveal small disturbances, where capillary waves discussed by Taylor (1959) are formed, which do not develop into holes. No doubt these are due to the impact of small particles which, if they pass through the sheet, produce holes so small that they close instead of opening out. That small holes would close when formed while

large ones would expand is to be expected because in both cases the surface area decreases. A sheet of uniform thickness t , containing initially a cylindrical hole of radius a for instance, would suffer a decrease in total surface area if the hole closed provided $a < t$, but if $a > t$ the surface area would be reduced when the hole opened out. This consideration suggests that it would be of interest to make an experimental study of the conditions under which holes close or open when the hole producing agent is withdrawn or passes through the sheet.

Experiments in which steel balls were projected through falling sheets of water failed to reveal any simple criterion dividing cases where the ball produced an expanding hole from those where the sheet reformed.

Though the idea that small holes will close and large holes open out resembles an unstable equilibrium situation there is no real analogy because no hole in a freely falling sheet can be in equilibrium. On the other hand, holes in a sheet of uniform thickness lying on a horizontal plane can be in equilibrium under the influence of gravity and surface tension, and such configurations have been described. Recently Padday (1971) has extended the range of such equilibrium calculations and has given a good list of references. Nothing seems to have been published on their stability. Apart from the mathematical difficulty of such stability calculations when the equilibrium configurations are so difficult to describe, one reason for the absence of literature on the subject may be that the angle of contact where an interface between two fluids meets a solid is not very easy to measure, and sometimes depends on whether the interface is advancing or retreating. Even in cases where a configuration might be unstable if the angle of contact were independent of other factors, it may be stabilized by dependence on the direction of motion of the interface in the same way that an inverted pendulum may be stabilized by friction of its pivots so that it can remain at rest upside down within a certain range of angles from the vertical. In experiments to be described holes were made in sheets of water and in mercury under water. In the former case the holes could remain and they were photographed (figure 11 for example), while in the latter the holes either closed or expanded as soon as they were formed and the diameter of the probe was recorded as in figure 10.

2. Equilibrium and stability of a soap film

If a hole were to exist in relative equilibrium in a freely falling thin sheet equilibrium would be maintained by surface tension forces. For an axisymmetric hole in a uniform sheet this would imply that the two principal curvatures are equal and provide equal and opposite surface tension stresses. Such an equilibrium is exactly that of a soap film suspended axisymmetrically, say on two circular rings. If the rings, of radius a , are spaced a distance $2d$ apart, $2d$ might be taken as analogous to the thickness of a sheet containing a hole which extends to a radius a .

The equation to the equilibrium configuration is well known (see for example Lamb 1916, art. 128), and is the catenary of revolution (or catenoid)

$$r/r_0 = \cosh(x/r_0), \quad (1)$$

where x and r are cylindrical polar co-ordinates, x measuring distance along the axis of symmetry and r the radial distance from the axis. The radius r_0 is the minimum radius, at the centre-plane $x = 0$. The form of this curve confirms that there is no equilibrium configuration for a hole in a plane free sheet. But for the film suspended on rings the boundary condition is

$$a/r_0 = \cosh (d/r_0), \tag{2}$$

and this may be taken as the equation determining r_0 . Writing $\xi = d/r_0$ equation (2) may be written in the form

$$\cosh \xi/\xi = a/d. \tag{3}$$

The function $\cosh \xi/\xi$ has its smallest possible value 0.152 when $\xi = \coth \xi = 1.200$. The greatest value of d/a at which a film can exist in equilibrium is therefore $d/a = 0.663$. This point is marked C in figure 1, which shows the catenary of equation (1).

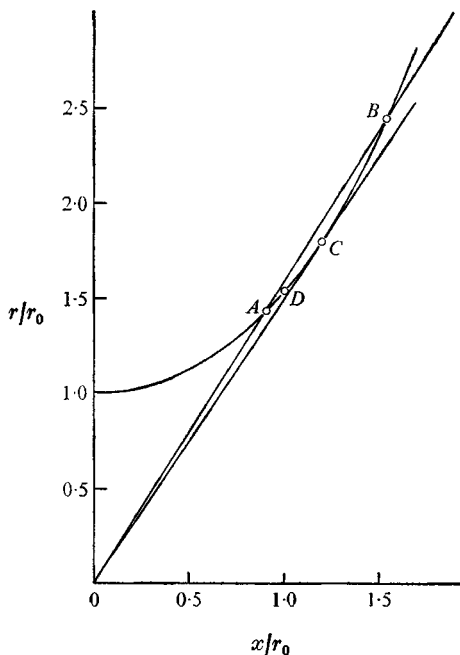


FIGURE 1. The catenoid solution: A and B give two solutions for $x/r = 0.625$; C is the critical point where $x/r = 0.663$; D represents the limiting position for a chain hanging symmetrically between two smooth pulleys.

It is interesting to compare this critical axial section with other critical catenary forms taken up by uniform chains hanging from fixed points. The catenary which spans the greatest horizontal length without raising the maximum stress beyond a given limit is the same as the critical catenary above (see Lamb 1916, example 3, p. 190). Another critical catenary is that taken by a chain of finite length when supported by two frictionless pulleys at the farthest distance apart consistent with equilibrium. The support points in that case are represented by D in figure 1, where $x/r_0 = 1$, and if l is the length of the chain the

greatest distance apart of the pulleys at which the chain can be in equilibrium is $l/e = 0.368l$.

For the equilibrium of the soap film, when $d/a < 0.663$ there are always two catenoids. One such pair is shown in figure 1 by the two points A and B , where the line $x/r = d/a = 0.625$ cuts the curve.

The potential energy of the suspended film is measured by its surface area S , which is given by

$$S = 2\pi d^2 \left\{ \frac{1}{\xi} + \frac{1}{2\xi^2} \sinh 2\xi \right\}. \quad (4)$$

For given d , $S(\xi)$ has a minimum at the point ξ_0 for which $\xi_0 = \coth \xi_0$. This is clearly the value of ξ at which the two equilibria coalesce. Thus the two equilibria are on opposite sides of the minimum of S .

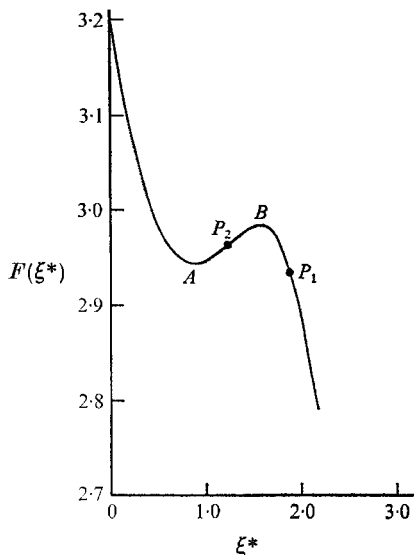


FIGURE 2. The variation of the surface energy $F(\xi^*)$, where $\xi^* = d/r_0^*$, for catenaries rotated about the axis of the rings. $d/a = 0.625$.

The stability of the film may be discussed in the context of the calculus of variations. It can be shown (for example as in Pars 1962, § 5.8) that the position A is stable to displacements of a general nature, and that B is unstable. We can verify directly that B is an unstable equilibrium by considering the variation of S with changes in the profile. An interesting family of membrane profiles to consider, which contains the given equilibria, is the family of the catenaries obtained by hanging uniform flexible chains of different lengths between the rings, and rotating about the axis of the rings. The surface area is found to be $2\pi d^2 F(\xi^*)$, where

$$F(\xi^*) = \left[\frac{1}{\xi^*} + 2 \frac{a}{d} \frac{\sinh \xi^*}{\xi^*} - \frac{1}{2} \frac{\sinh 2\xi^*}{\xi^{*2}} \right] \quad (\xi^* = d/r_0^*),$$

and r_0^* is the parameter of the catenary. This area is the same as (4) when $\xi^* = \xi_A$ or ξ_B , and it is easily verified that $F(\xi^*)$ has a maximum at $\xi^* = \xi_B$ and a minimum at $\xi^* = \xi_A$. The function $F(\xi^*)$, when $d/a = 0.625$, is shown in figure 2. The maximum of $F(\xi^*)$ at B proves that B is an unstable equilibrium.

For smooth small displacements a weak minimum of A may be proved using the theory of the second variation. Let the membrane incur a small displacement so that the radius is changed to $r(x) + \Delta(x, \theta)$, where $\Delta(x, \theta)$ is a smooth function of x and the azimuth angle θ , which is zero at the rings. The change in S is zero to the first order in Δ , and is given to the second order by

$$\delta S = \frac{1}{2} \int_0^{2\pi} \int_{-d}^{+d} \left\{ r_0 \Delta_x^2 - \frac{1}{r_0} \Delta^2 \right\} \operatorname{sech}^2 \frac{x}{r_0} + \frac{\Delta_\theta^2}{r_0} \Bigg\} dx d\theta,$$

where $\Delta_x = \partial\Delta/\partial x$ and $\Delta_\theta = \partial\Delta/\partial\theta$, and these derivatives are taken as small quantities of the same order as Δ . Since the last term in the integrand is positive, to show the stability of A we need to ensure that

$$\int_0^{2\pi} \int_{-d}^{+d} \{ r_0 \Delta_x^2 - (1/r_0) \Delta^2 \} \operatorname{sech}^2 (x/r_0) dx d\theta \tag{5}$$

is positive definite. This can be seen by writing (5) in the form

$$\frac{1}{2} \int_0^{2\pi} \int_{-d}^{+d} u^2 r_0 \left[\frac{\partial}{\partial x} \left(\frac{\Delta}{u} \right) \right]^2 \operatorname{sech}^2 \frac{x}{r_0} dx d\theta, \tag{6}$$

where $u = \cosh (x/r_0) - (x/r_0) \sinh (x/r_0)$, and $u > 0$ for $|x| \leq d$ when $\xi < \xi_A$. A strong minimum of S (in which Δ_x and Δ_θ are not assumed small) can also be proved at $\xi = \xi_A$ by appeal to the Weierstrass sufficiency theorem (Pars 1962). Since

$$S = 2\pi \int_{-d}^{+d} r(1+p^2)^{\frac{1}{2}} dx,$$

where $p = dr/dx$, to show this it suffices that $\partial^2\{r(1+p^2)^{\frac{1}{2}}\}/\partial p^2 = r(1+p^2)^{-\frac{3}{2}}$ is positive, and that there is no conjugate point.

An alternative method of discriminating between the stable and unstable positions of equilibrium is to consider the energy balance when the rings are displaced so that d changes to $d + \delta d$ without changing a . The force exerted by each ring in the x direction is $4\pi a T \operatorname{sech} \xi$ in equilibrium, so that

$$8\pi a T \operatorname{sech} \xi \cdot \delta d = 2T(\partial S/\partial d) \delta d.$$

The stability is determined by the sign of

$$\frac{\partial}{\partial d} \left(\frac{\partial S}{\partial d} - 4\pi a \operatorname{sech} \xi \right)$$

at the equilibrium position. On differentiating, this quantity is found to be $(4 \cosh^2 \xi \sinh^3 \xi)/(\cosh \xi - \xi \sinh \xi)^3$. When $\xi = \xi_A$ it is positive and when $\xi = \xi_B$ it is negative. Thus A is a stable position and B an unstable position.

Figure 3 shows the variation of r_0/a with d/a for the catenoid solution. For given a , r_0 increases with increasing d for the unstable position, and decreases for the stable position. If we regard the membrane on rings as simulating a hole in a sheet these results lend support to the conjecture that there is a critical hole size above which a hole will open, and below which it closes. Referring to figures 2 and 3, a hole with minimum radius r_0^* corresponding to a point P_1 one would expect to close. A hole corresponding to P_2 should expand towards the stable equilibrium

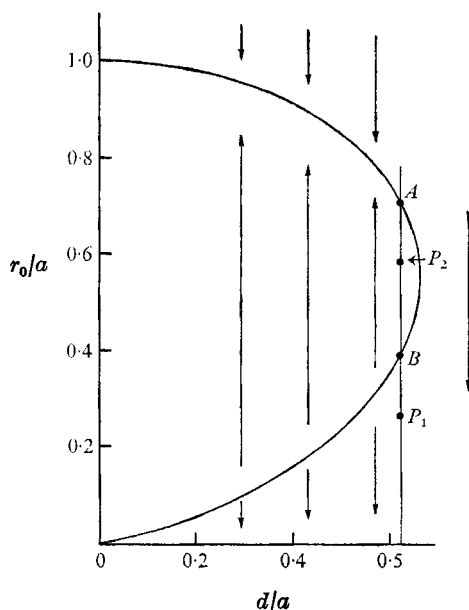


FIGURE 3. The variation of r_0 with d for the catenoid solution. Arrows indicate the direction in which the minimum radius might be expected to change.

position. This suggests that the critical hole size is given by the value of r_0 at the unstable equilibrium B for any given d . As d/a increases towards the critical value 0.663, the two equilibria come closer together until they coalesce at the critical point. Furthermore, when the stable equilibrium reaches its critical state, the form of figure 2 as $\xi_A \rightarrow \xi_B$ suggests that the film will then collapse into the axis.

3. Experiments with catenoid soap films

The apparatus used for producing catenoid soap films is shown in figure 4. It consists of two copper plates A and B , each containing a hole 3.0 cm diameter. A is soldered to an upright post C on which B can slide so that it remains parallel to A . A second post D which is fixed to A serves as a guide to keep the holes coaxial. A stop E can be fixed on C so that the plate B can rapidly be moved to a predetermined position. The plates A and B are put together into a dish of soap solution and B is then raised till the guide F strikes the stop E . It is then found that a complicated film system is formed, but by destroying the films which form between the guide posts, the film shown in figure 4 and photographed in figure 5 (plate 1) is formed. The central plane film G is then destroyed and the catenoid is left. The stop E can then be moved out to near the theoretical position for maximum plate separation, namely $3 \times 0.66 = 1.99$ cm. In fact, the film collapsed when the separation was 2.05 cm but remained coherent at 1.99 cm. The photograph in figure 6 (plate 1) shows this film.

In the configuration shown in figures 4 and 5, the catenoid sections of the film meet the central plane section G at angles of 120° . Using this fact, it can be seen

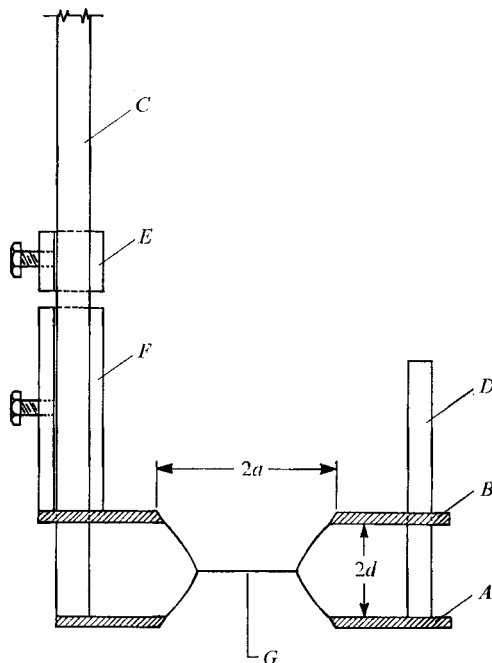


FIGURE 4. Schematic diagram of an apparatus used for producing soap films.

that the limiting position of equilibrium is when $d/a = 0.4078$, which gives a maximum separation of 1.223 cm. The film collapsed when $2d = 1.27$ cm but was stable when $2d = 1.22$ cm.

4. Equilibrium and stability of holes in an infinite sheet lying on a plane

For holes in a sheet whose diameter is of the order of the depth h gravity forces become comparable with surface tension forces when $g(\rho_1 - \rho_2)h^2/T$ is of order unity, where ρ_1 and ρ_2 are the densities of the heavier and lighter fluids forming the hole surface respectively, T is the surface tension and g is the acceleration of gravity. The equilibrium configuration of a hole in an infinite sheet lying on a plane depends on three variables, the depth h , the minimum radius R and the angle of contact θ . The calculations can be expressed in non-dimensional form by dividing all the lengths by the length $k = [T/g(\rho_1 - \rho_2)]^{1/2}$. In this form the equation of equilibrium of the hole becomes

$$\frac{r''}{(1+r'^2)^{3/2}} - \frac{1}{r(1+r'^2)^{1/2}} = x_1 - x, \tag{7}$$

where $r' = dr/dx$ and $r'' = d^2r/dx^2$. As shown in figure 7 the plane $x = 0$ is taken at the level of the minimum radius R , and $x = x_1$ is the upper plane surface of the sheet. It is interesting to note that the addition of the gravity force on the right-hand side of (7) does enable a solution to be found which has one asymptote at $x = x_1$. If $\eta = x_1 - x$, as $\eta \rightarrow 0$, r , r' and r'' all become large, and the first term of (7),

representing the curvature in the meridian plane, dominates on the left-hand side. Asymptotically this term balances the gravity force. Thus for small η ,

$$d^2r/d\eta^2 \sim \eta(dr/d\eta)^3,$$

so that $r \sim -\log \eta$ as $\eta \rightarrow 0$.

Studies of the meniscus equation (7) have been made by many workers, and a detailed account of previous work is given in the comprehensive study by Padday (1971). More recent results given by Padday & Pitt (1972) give 'rod in free surface profiles' which correspond to the profiles of holes in this context when $x > 0$. For our purposes it is desirable to correlate the values of R and h , for a given value of θ .

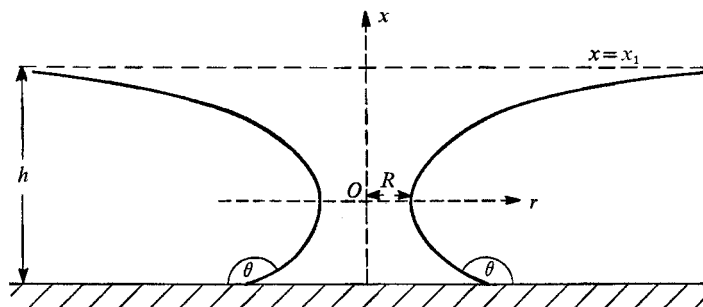


FIGURE 7. Diagram giving notation for the analysis of the equilibrium of a hole in a fluid layer supported on a horizontal plane.

Solutions of (7) were calculated numerically, using the gradient angle ψ , where $\tan \psi = dr/dx$. It is convenient to rewrite the equation in terms of an auxiliary parameter t , in the form

$$\left. \begin{aligned} dr/dt &= r \sin \psi, \\ dx/dt &= r \cos \psi, \\ d\psi/dt &= (x_1 - x)r + \cos \psi. \end{aligned} \right\} \quad (8)$$

Numerical integration of equations (8) was performed using a fourth-order Runge-Kutta procedure in a program written by Miss A. E. Latham. A value of $r = R$ was assigned at which $x = 0$ and $\psi = 0$. By running trial values of x_1 , this quantity could be found with sufficient accuracy by obtaining a lower bound for which $\psi < \frac{1}{2}\pi$ when $x = x_1$, and an upper bound for which ψ exceeds $\frac{1}{2}\pi$ before $x = x_1$ is reached. Step lengths of 0.001 in t were found sufficient to establish x_1 to two places of decimals for values of R down to 0.1. Having established x_1 , the equations can then be integrated backwards in x to obtain the height of the layer below $x = 0$, down to the point $\psi = \frac{1}{2}\pi - \theta$ (< 0), which represents the contact with the plane. Thus the total height of the sheet could be ascertained for given θ and R . Figure 8 gives the results of these calculations, in which the loci of equilibria in the h, R plane are given for a range of contact angles. The results given for $\theta = 90^\circ$ are in close agreement with values calculated from table 3 of Padday & Pitt for 'rod in free surface profiles', which are marked in figure 8 with an \times . It is seen that R increases monotonically with h and approaches

infinity as h approaches an asymptotic value. The asymptote may easily be ascertained since it represents the value of h for a plane hole. In this case the second term of the left-hand side in (7) is absent and the equation can be integrated in simple terms. The solution is given by Lamb (1916, art. 127), from which it can be verified that the asymptotic value of h is $2 \sin \frac{1}{2}\theta$.

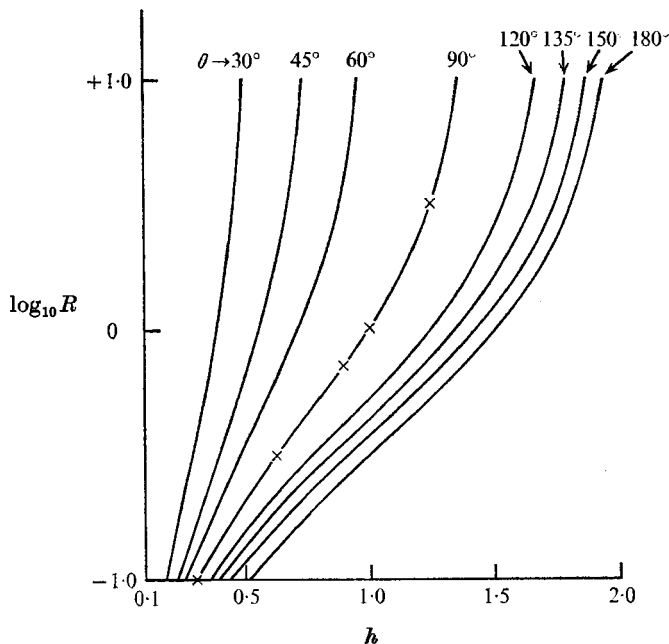


FIGURE 8. Numerical calculation of minimum hole radius R as a function of height h , for fixed contact angle θ . R and h are made dimensionless with respect to k .

Figure 8 is analogous to figure 3 for the catenoid solution, and it may be conjectured from the analogy that the equilibria obtained here are unstable. To establish this instability it suffices to show that there is one small displacement for which the potential energy is reduced.

Consider the mass of fluid in the sheet within a large radius $r = L$. We construct an axisymmetric virtual displacement of this fluid which (a) is incompressible, (b) keeps the angle of contact θ unchanged, and (c) moves each element horizontally in the radial direction of r . To satisfy these conditions let the radius $r(x)$ at the hole surface change to $r(x) - \delta$, where δ is a small constant independent of height x . Evidently this gives rise to a radially inward displacement δ^* at the radius r^* ($> r$) at the same height, where $r^2 - (r - \delta)^2 = r^{*2} - (r^* - \delta^*)^2$. Since gravity does no work on the fluid the work done is

$$\pi(\rho_1 - \rho_2)g \int_{x_2}^{x_1(L)} (x_1 - x) (2\delta r - \delta^2) dx + \pi T \cos \theta \{2r_2 \delta - \delta^2\} - \pi T \sin \psi_1(L) \{2L\delta - \delta^2\}.$$

Here r_2 and x_2 represent the values of r and x at the base, $x_1(L)$ is the value of x at the upper surface at $r = L$, and $\psi_1(L)$ is the slope angle at this point. The surface

area increases by $-2\pi\delta[s]_{x_2}^{x_1(L)}$, where $[s]_{x_2}^{x_1(L)}$ denotes the arc length of the hole profile for the section considered. A sufficient condition for instability will be

$$\pi(\rho_1 - \rho_2)g \int_{x_2}^{x_1(L)} (x_1 - x)(2\delta r - \delta^2) dx + \pi T \cos \theta (2r_2 \delta - \delta^2) - \pi T \sin \psi_1(L) (2L\delta - \delta^2) + 2\pi T [s]_{x_2}^{x_1(L)} \delta > 0,$$

to order δ^2 , the terms of order δ being zero. After some reduction we find this quantity to be

$$\delta^2 \left[\pi(\rho_1 - \rho_2)g \left\{ \frac{(x_1 - x)^2}{2} \right\}_{x_2}^{x_1(L)} + \pi T \{ \sin \psi_1(L) - \cos \theta \} \right],$$

which becomes in the limit, as $L \rightarrow \infty$, $\frac{1}{2}\delta^2\pi T(4 \sin^2 \frac{1}{2}\theta - h^2)$. Since $h < 2 \sin \frac{1}{2}\theta$ the equilibrium is statically unstable. The virtual displacement used here is a sliding displacement at the base of the sheet, and so would not represent the actual movement of the layer away from the equilibrium, but the virtual displacement could be made more realistic, say by excluding from the displacement a very thin layer adjacent to the plane, without invalidating the result.

We may interpret figure 8 in the same way as figure 3 for the catenoid. For $h = h_1 < 2 \sin \frac{1}{2}\theta$, the value of R for the unstable equilibrium should represent a critical hole size, for given θ , in an infinite sheet. Larger holes should open out, and smaller ones should close up.

5. Experiments on making holes in a mercury sheet

Holes were made in a mercury sheet below water using a graduated series of accurately made cylindrical probes. With probes of small diameter the holes closed up when the probe was removed, but as probes of increasing diameter were tried a critical diameter was found above which the hole opened out. The sheets were contained in an accurately made perspex (lucite) dish, with internal diameter 14.2 cm and depth 5 cm (see figure 9), on to the bottom of which a glass disk was fixed centrally. In the first experiments this disk was 4 cm in diameter and 0.1 cm thick, but this was so thin that the fluid left the wall of the dish while it was still continuous over the disk. The disk was replaced in later experiments by one 0.635 cm thick \times 5.08 cm diameter, and over this disk holes in mercury which opened could still be made when the diameter of the hole maker was 0.16 cm and the depth of mercury 0.19 cm.

Two brass bridges, which could be accurately located by means of pins which mated with holes in the rim of the dish, carried respectively a micrometer for measuring the depth at the centre, and a steel capillary tube 0.16 cm in diameter as shown in figure 9. This tube served as a guide to the probes when making holes and as a means of introducing the upper fluid at the bottom of the probe by a syringe so that the fluid which was being penetrated did not flow back when the probe was lifted off the central glass disk. As the probes were carefully lifted off by hand the correct manipulation of this syringe proved to be a critical feature of the experiments with mercury.

It was realized that the dish, which had been designed for another purpose, was too small to obtain results for large holes which could usefully be compared with

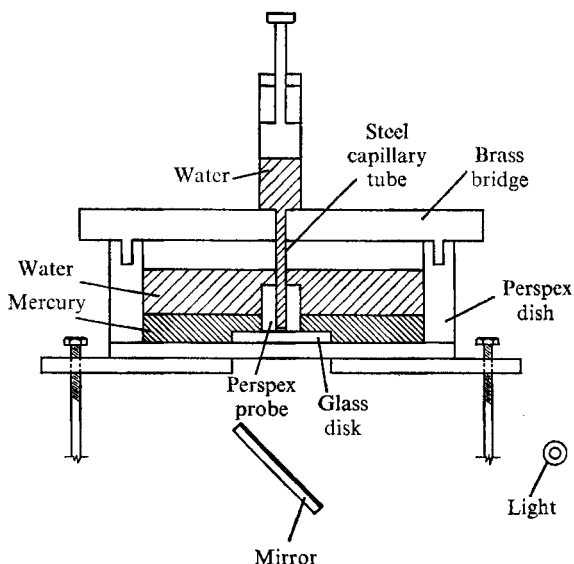


FIGURE 9. Diagram of apparatus used for producing holes in a layer of mercury under water. The same apparatus was used to produce holes in a water sheet using an air jet instead of the probes, when the disk was coated with paraffin wax.

Probe diameters (cm)													
0.502	0.580	0.672	0.752	0.821	0.897	1.016	1.139	1.403	1.703	2.109	2.302	2.504	2.554
Sleeve diameters (cm)													
0.247	0.296	0.347	0.394	0.449									

TABLE 1. Probe and sleeve sizes

calculations which assumed that the fluid extended to a very large area, because the depth was measured after the hole had been closed. The increase in depth when a hole expands in a dish of limited diameter must have a stabilizing effect on the expanding hole, and in fact within a small range of depths a hole did sometimes expand after being formed to a radius which was still within the area of the glass disk. If the hole diameter is $2R$ cm the general level of the fluid may be expected to be reduced approximately by a factor $(2R/14.2)^{\frac{1}{2}}$ when the hole closes in a dish 14.2 cm in diameter.

The diameters of the perspex probes used are listed in table 1. Each had a central hole slightly larger than 0.16 cm, and they were short enough to be completely immersed in the upper fluid when they reached the lower fluid. For the smaller holes a set of steel sleeves was used instead, whose dimensions are also given in table 1.

Figure 10 shows the results of the experiments in which a mercury sheet under water was penetrated by probes. A circle represents an experiment in which the

hole closed as the probe was removed and a cross one where it opened out. It will be seen that at a given depth probes larger than a critical diameter open up a hole, but smaller probes make holes which collapse when the probes are removed.

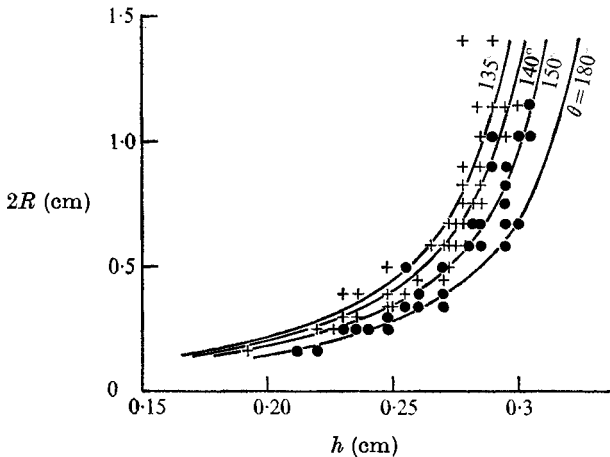


FIGURE 10. Experimental observations of holes in a mercury sheet with theoretical equilibrium curves superposed. ●, hole which closed; +, hole which opened out.

The fact that there appeared to be a sharply defined critical probe size dividing those which made holes which closed when the probe was removed from those which opened suggested that, unlike the water experiments to be described later, changes in the angle of contact with direction of motion of the meniscus had little stabilizing effect in the case of holes in mercury sheets under water. The surface tension of the interface between water and mercury has been given values ranging from 350 to 400 dynes/cm in the literature, but recently Padday & Pitt† have redetermined it using all the resources of their laboratory and computer, and their value is 384.4 dynes/cm. With this value of T and with $\rho_1 = 13.6 \text{ g/cm}^3$ and $\rho_2 = 1 \text{ g/cm}^3$ $k = 0.176 \text{ cm}$. Direct measurement of the angle of contact is difficult to achieve. For purposes of comparison the curves of (h, R) of figure 8 for $\theta = 135^\circ$, 140° , 150° and 180° are shown in figure 10 using $k = 0.176 \text{ cm}$. Good agreement with the observations is given when θ is 150° .

6. Experiments with air holes in a water sheet

To make holes in a sheet of water lying on a horizontal surface that surface must be hydrophobic and to make the glass disk hydrophobic a solution of pure paraffin wax in a volatile fluid known as petroleum ether was run over the glass disk and allowed to dry. The wax coating thus produced turned out to be very uniform but it was liable to be damaged when the probes used in the mercury experiments were used. Holes could be made, however, without damaging the wax coating by blowing an air jet vertically downwards on to the surface of the water. Figure 11 (plate 2) gives photographs of two of the holes produced in this

† Private communication.

way which remained open after the blowing had stopped. It will be seen that they are good circles indicating that the angle of contact round the edge, which one would expect to depend on the quality of the deposit of wax, is constant. Holes could be made with the probes but their irregularities indicated that the wax coating had been spoiled. The experimental method was to use the same micrometer mounted on a bridge as was used in the mercury experiments for measuring the depths, but the diameters of the holes were measured by photographing them from above by light reflected from a mirror placed below the tank. The source of light was an electric bulb about 4 ft away. The procedure was to measure the depth, then blow holes of increasing size till one remained open when the air jet was cut off. Larger holes would remain open and within a small range of depths a large hole would increase when the air jet was cut off. The fact that holes can remain stationary is contrary to the predictions from the theory given earlier for an invariant angle of contact, and it suggests that the angle of contact depends on whether the meniscus is advancing, in which case the hole is closing, or retreating, when the hole is opening. If this is so the angle of contact θ_a for the advancing meniscus will be larger than the value θ_r for a retreating meniscus. If the curves for these two angles of contact are drawn in the h, R plane of figure 8 they define a range of R for given h within which a hole can be trapped.

The fact that the holes made in these experiments can remain at rest over a large range of sizes must be due to a large difference between θ_a and θ_r . We are grateful to one of the referees for pointing out that steady dynamical measurements of θ_a and θ_r for water on paraffin wax were performed by Ablett (1923), using a partly immersed rotating cylinder coated with the wax. Ablett found that the contact angles were identical at steady slow rotations with $\theta = 104^\circ 34'$ and at higher velocities they reached constant, but differing values $\theta_a = 113^\circ 9'$ and $\theta_r = 96^\circ 20'$. Here it was not found possible to measure the contact angles directly, and indeed the contact angle often takes time to establish itself at its final value, so that even if a final value attained after a long period of observations could be measured it would not be useful in making comparisons between theory and experiments with rapidly made holes. One method for measuring contact angles of a water/air interface where it meets a solid surface is to push a capillary tube of the solid material into water and observe the height of the meniscus above or below the level of the surrounding water.

If d is this height and a ($\ll d$) is the radius of the capillary tube

$$2\pi aT \cos \theta = \pi a^2 \rho g d,$$

so that

$$\cos \theta = \frac{1}{2}(\rho g a d / T). \quad (9)$$

To prepare a surface which would be comparable with that used in the experiments a glass capillary tube of 0.1 cm bore was filled with a solution of paraffin wax in petroleum ether of the same concentration as that used in the hole-making experiments. This was then emptied and dried. When the tube was pushed into water the meniscus rapidly rose till it was 1.45 cm below the surface of the surrounding water. It was then left in position. The meniscus rose very slowly till after some hours it had risen to 1.24 cm below the water level. The tube was then raised and the meniscus fell rapidly to a level 1.25 cm above the surrounding

water. After some hours it had fallen to 1.15 cm above the level outside. Taking $T = 74$ dynes/cm (9) gives $\theta_a = 119^\circ$ for $d = -1.45$ cm and $\theta_r = 65.7^\circ$ for $d = 1.25$ cm. To find whether the thickness of the wax film was critical the film was deposited from wax solutions of different concentrations and the results differed only very slightly. It will, however, be noticed that when θ is less than 90° the radius of the line of contact of the edge of the hole is slightly greater than that which has been defined as R . To make a comparison with theory the dependence of the radius of the ring of contact on the depth of the fluid has been calculated for the case when $\theta = 65^\circ$. It will be seen (figure 12) that very little error would be committed if the observed radius of the hole were taken as that at which tangents can be vertical.

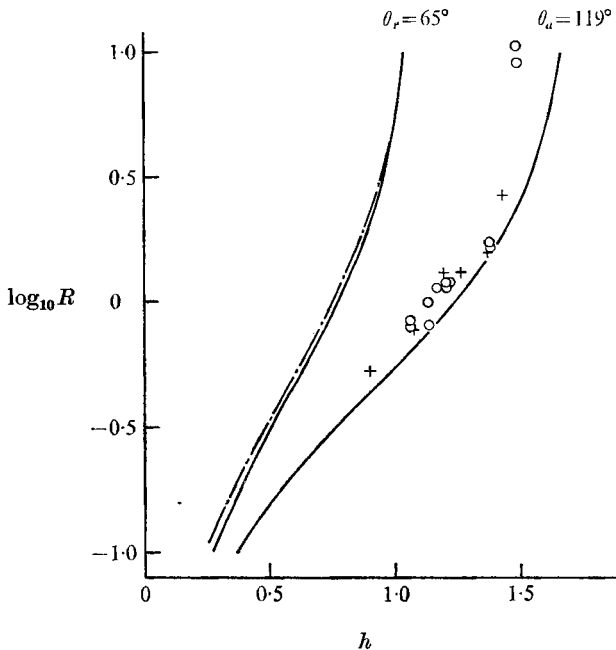


FIGURE 12. Experimental observations and predictions for holes in water on paraffin wax: —, R for $\theta = 65^\circ$ and 119° ; ---, the minimum radius for $\theta = 65^\circ$; O, +, two sets of observations of the minimum hole size which would remain open.

The results of two sets of measurements made with two glass disks coated with wax at different concentrations are shown in figure 12. The value of k used to make the observed measurements non-dimensional is $k = 0.274$ cm. It will be seen that the smallest hole that can be made at any given depth is always slightly greater than the value calculated for an advancing meniscus with $\theta_a = 119^\circ$. This could be expected in any case because the method of blowing, of gradually increasing the size till a hole remained open, was not mechanized and required a little practice.

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